American Journal of Optics and Photonics
2018; 6(3): 31-41
http://www.sciencepublishinggroup.com/j/ajop
doi: 10.11648/j.ajop.20180603.12
ISSN: 2330-8486 (Print); ISSN: 2330-8494 (Online)

Higher-Order Nonlinear Schrödinger Equation Family in Optical Fiber and Solitary Wave Solutions

Rodrique Njikue[1, 3], Jean Roger Bogning[2, *], Timoleon Crépin Kofané[1, 3]

[1]Department of Physics, Faculty of Science, University of Yaounde I, Yaoundé, Cameroon
[2]Department of Physics, Higher Teacher Training College, University of Bamenda, Bamenda, Cameroon
[3]African Center of Excellence in Information Technology and Telecommunications, University of Yaoundé I, Yaoundé, Cameroon

Email address:
rodriquenjikue@yahoo.fr (R. Njikue), rbogning@yahoo.com (J. R. Bogning), tckofane@yahoo.com (T. C. Kofané)

[*]Corresponding author

To cite this article:
Rodrique Njikue, Jean Roger Bogning, Timoleon Crépin Kofané. Higher-Order Nonlinear Schrödinger Equation Family in Optical Fiber and Solitary Wave Solutions. *American Journal of Optics and Photonics*. Vol. 6, No. 3, 2018, pp. 31-41. doi: 10.11648/j.ajop.20180603.12

Received: August 19, 2018; **Accepted**: September 6, 2018; **Published**: December 14, 2018

Abstract: In this paper, we modify with an appropriate analytical technique, the characteristics of the optical fiber through the modification of the coefficients of the highly nonlinear partial differential equation, which initially governs the dynamics of the propagation in such a wave guide. The procedure consists to assign arbitrary coefficients to the various terms of the established nonlinear partial differential equation, such as the one that embodies the propagation dynamics in a strongly nonlinear optical fiber and subsequently establishing the constraint equations linking these coefficients and thus the analys is makes it possible to enumerate the criteria for which obtaining the desired solutions is possible. These coefficients are like indicators which characterize the various modifications made in this medium of transmission. The nonlinear evolution equation that served as mathematical model for this study is the higher-order nonlinear Schrödinger equation which better describes the propagation of an ultrafast pulse in an optical fiber. The use of the Bogning-Djeumen Tchaho-Kofané method enabled not only to establish the constraint relations, but also the solitary wave solutions and plane wave solutions. We want through the results obtained in this article to give the specialists of the manufacture of transmission media such as optical fiber, to consider the modification of the properties of this wave guide during manufacture, depending on the type of signal that one wants to propagate in this case notably the solitary wave.

Keywords: Schrödinger Equation, Higher-Order Nonlinear Effects, Solitary Wave Solution, Periodic Travelling Wave Solutions, Bogning–Djeumen–Kofané Method

1. Introduction

Nonlinear Schrodinger (NLS) equation is one of the most interesting nonlinear evolution equation (NEE) used to model phenomena in many area of physics in which nonlinear optics, plasma physics, condensed matter physics, nonlinear quantum field theory, bio-physics and hydrodynamic [1-8]. Particularly in nonlinear optics, the propagation of picoseconds pulse in monomode fibers is governed by the well-known NLS equation. This equation is known to be integrable and admits bright and dark soliton solutions in anomalous dispersion region and normal dispersion regime respectively [9-10]. For femtosecond pulse propagation, the standard NLS equation is inappropriate. Thin width pulses will induce higher-order effects like third order dispersion (TOD) and nonlinearity such as self-steepening (SS) and stimulated Raman scattering (SRS) [11-12]. The governing equation of the femtosecond pulse propagation in monomode fibers is the Higher-order Nonlinear Schrödinger (HNLS) equation [13]

$$\begin{aligned} &E_z - i\alpha_1 E_{tt} - i\alpha_2 |E|^2 E - \alpha_3 E_{ttt} \\ &-\alpha_4 \left(|E|^2 E\right)_t - \alpha_5 \left(|E|^2\right)_t E = 0 \end{aligned} \tag{1}$$

Where E is the slowly varying envelope of the electricfield, α_1 and α_3 the dispersion coefficients, α_2, α_4 and α_5 the nonlinear coefficients.

In nonlinear sciences, one thing is to model phenomena using NEEs, another thing is to construct their exact solutions. The survey of these exact solutions of nonlinear evolution equations has a great importance in the study of nonlinear phenomena. In optical fiber for example, they permit to understand better, phenomena like soliton propagation and soliton interactions, and their Modulational instabilities [11, 14-17]. Several methods have been proposed to solve NLS equation of higher-order, we can quote the inverse scattering transform, Galilean transformation, Hirota's bilinear method, Liouville integrability and Gauge equivalence, the unified transform method, the Sub-ODE method [18-22]. Thus many type of solutions have also been found, such as dark and bright solitary wave solutions, Jacobi elliptic function solutions, and rotational solutions [23-24].

The goal in this study on the one hand is to construct optical solitary wave solutions and periodic travelling wave solutions (PTWS) of the generalized higher-order nonlinear Schrödinger (GHNLS) equation

$$\mathrm{n}_0 E_z + i\mathrm{n}_1 E_{tt} + i\mathrm{n}_2 |E|^2 E + \mathrm{n}_3 E_{ttt} + \mathrm{n}_4 \left(|E|^2 E\right)_t + \mathrm{n}_5 \left(|E|^2\right)_t E = 0 \tag{2}$$

Where n_0 represents coefficient related to the variation of the electricfield with respect to z, n_1 and n_3 the group velocity dispersion coefficient and the TOD coefficient, n_2, n_4 and n_5 respectively the coefficient related to the self-phase modulation, SS and SRS.

To reduce the density of work in this study, we set

$$E(z,t) = \overline{E}(z,t)e^{-i(kz-\omega t)} \tag{3}$$

Substituting Eq.(3) into Eq.(2), this last equation is written asfollows

$$-i(k\mathrm{n}_0 + \mathrm{n}_1\omega^2 + \mathrm{n}_3\omega^3)\overline{E} + i(\mathrm{n}_1 + 3\mathrm{n}_3\omega)\overline{E}_{tt} + i(\mathrm{n}_2 + \mathrm{n}_4\omega)\overline{E}^3 + (-2\mathrm{n}_1\omega - 3\mathrm{n}_3\omega^2)\overline{E}_t + \mathrm{n}_0\overline{E}_z + \mathrm{n}_3\overline{E}_{ttt} + (3\mathrm{n}_4 + 2\mathrm{n}_5)\left|\overline{E}\right|^2\overline{E}_t = 0 \tag{4}$$

The mathematical method employed to obtain the results is the Bogning-Djeumen Tchaho-Kofané method (BDKm) [25-32]. The BDKm is a method proposed by J. R. Bogning, C. T. Djeumenand T. C. Kofané in 2012 to construct exact solutions of certain NEEs of the form

$$a_i \sum_i (\frac{\partial q}{\partial x_i})_i + b_i \sum_i (\frac{\partial^2 q}{\partial x_i^2}) + ... + c_i \sum_i (\frac{\partial^l q}{\partial x_i^l}) + d_i \sum_{m,n} (\frac{\partial^n q \partial^m q}{\partial x_i^n \partial x_i^m}) + f(q, |q|^2) = 0 \tag{5}$$

Where a_i, b_i, c_i and d_i are constants that characterize partial differential equations, i, l, m and n positive natural integers, f an arbitrary linearity function of q and $|q|^2$, and q the unknown function to determine and $|q|$ its magnitude.

A brief presentation of the novel BDKm is also important for the calculations made.

2. Description of the Method

To research exact solutions of Eq.(5), the main steps are the following:

Step 1: Setting the change of variable

$$q(z,t) = q(\xi), \xi = \alpha\left(z - \beta t\right) \tag{6}$$

Where β is a real number and α a real or pure imaginary constant which will be consider in a first time as real, Eq.(5) is transformed into and ordinary differential equation

$$P(q, q', q'', ..., |q|^2 q', q(|q|^2)'...) = 0 \tag{7}$$

where $q' = \frac{\partial q}{\partial \xi}$

Step 2: Assuming that solution of Eq. (7) can be expressed under the form:

$$q(\xi) = a_1 \sinh^m(\xi)\sec h^n(\xi) + ... + a_l \sinh^q(\xi)\sec h^r(\xi) \tag{8}$$

Where m, n, q and r are numbers to determine, Eq. (8) into Eq. (7), and taking into account the transformations related to the BDKm [30], gives an equation of the form

$$\begin{aligned}&\sum_{i,j} F(a_i,\alpha,\beta,\omega,k)\cosh^j(\xi) + \sum_{i,k} H(a_i,\alpha,\beta,\omega,k)\mathrm{sech}^k(\xi) + \sum_{i,s} G(a_i,\alpha,\beta,\omega,k)\sinh(\xi)\sec h^s(\xi)\\ &+\sum_{i,t} T(a_i,\alpha,\beta,\omega,k)\sinh(\xi)\cos h^t(\xi) + \sum_i V(a_i,\alpha,\beta,\omega,k) = 0\end{aligned} \tag{9}$$

Step 3: If the functions F(a$_i$, α, β, ω, k), H(a$_i$, α, β, ω, k), G(a$_i$, α, β, ω, k), T(a$_i$, α, β, ω, k), V(a$_i$, α, β, ω, k), can be null for some values of the coefficients a_i, α, β, ω or k (nontrivialvalues), the ansatz given in Eq. (8) can be supported by Eq. (5) for any values of $m, n, \ldots, q, r$ and can be considered as solution of Eq. (5).

Step4: For $m = n = 0$, and $q = r$, we look for the values of r which can make some terms of Eq.(9) merge. The values of

r (r_{min}≤r≤r_{ax}) obtained permit to select the reals m,n,q that can gives solutions of Eq. (5).

Step5: For each values of $(m,n,...,q,r)$ selected between the previous values, we obtain from Eq. (9) a range of equations in a_i, α, β, ω and k to determine. Finally, the solitary wave solutions andother types of exact solutions (traveling wave solutions, hyperbolic function solutions…) are obtained.

Lemma: when the values $(m,n,q,r=l)$ give a trivial solution of Eq. (5), values m, n, q, $(m,n,q,r=l+1)$ (if l >0) and $(m,n,q,r=l-1)$ (if l <0) cannot give a nontrivial solution.

Step6: After getting the solutions when α is a real number (hyperbolic function solutions and solitary wave solutions), we consider α as a pure imaginary number by setting α=iμ where μ is areal. The insertion of the new values of α in Eq. (9), permits to write all PTWS from hyperbolic function solutions through the transformations

$$\begin{aligned}&\sinh(i\xi)=i\sin(\xi),\cosh(i\xi)=\cos(\xi)\\&\mathrm{sech}(i\xi)=\sec(\xi),\tanh(i\xi)=i\tan(\xi)\end{aligned}\tag{10}$$

3. Determination of the Coefficient Range Equations

Setting the travelling wave transformation

$$\overline{E}(z,t)=\overline{E}(\xi),\xi=\alpha(t-\beta z)\tag{11}$$

Eq. (2) becomes

$$\begin{aligned}&-\alpha(\beta \mathrm{n}_0+2\mathrm{n}_1\omega+3\mathrm{n}_3\omega^2)\overline{E}_\xi+\mathrm{n}_3\alpha^3\overline{E}_{\xi\xi\xi}+\alpha(\mathrm{n}_4+\mathrm{n}_5)\left(\left|\overline{E}\right|^2\right)_\xi\overline{E}+\alpha\mathrm{n}_4\left|\overline{E}\right|^2\overline{E}_\xi\\&+i[-(k\mathrm{n}_0+\mathrm{n}_1\omega^2+\mathrm{n}_3\omega^3)\overline{E}+\alpha^2(\mathrm{n}_1+3\mathrm{n}_3\omega)\overline{E}_{\xi\xi}+(\mathrm{n}_2+\mathrm{n}_4\omega)\overline{E}^2E]=0\end{aligned}\tag{12}$$

To look for the solution of Eq. (11), let consider the solitary ansatz

$$\overline{E}(t)=a\sinh^m\alpha(t-\beta z)\sec h^n\alpha(t-\beta z)+b\sinh^q\alpha(t-\beta z)\sec h^r\alpha(t-\beta z)\tag{13}$$

Where a and b are real constants, α the pulse width and β the shift of the inverse velocity group. Taking into account the fact that $\overline{E}$ is a real physical size, Eq. (12) can be separate in real part and imaginary part as follows

$$\alpha(\beta\mathrm{n}_0-2\mathrm{n}_1\omega-3\mathrm{n}_3\omega^2)\overline{E}_\xi+\mathrm{n}_3\alpha^3\overline{E}_{\xi\xi\xi}+\alpha(3\mathrm{n}_4+2\mathrm{n}_5)\overline{E}^2\overline{E}_\xi+i[-(k\mathrm{n}_0+\mathrm{n}_1\omega^2+\mathrm{n}_3\omega^3)\overline{E}+\alpha^2(\mathrm{n}_1+3\mathrm{n}_3\omega)\overline{E}_{\xi\xi}+(\mathrm{n}_2+\mathrm{n}_4\omega)\overline{E}^2E]=0\tag{14}$$

Eq. (14) can also be written

$$\alpha\kappa_1\overline{E}_\xi+\alpha\kappa_2\overline{E}^2\overline{E}_\xi+\alpha^3\kappa_3\overline{E}_{\xi\xi\xi}+i\kappa_4\overline{E}+i\kappa_5\overline{E}^3+i\alpha^2\kappa_6\overline{E}_{\xi\xi}=0\tag{15}$$

where,

$$\kappa_1=-\beta\mathrm{n}_0-2\mathrm{n}_1\omega-3\mathrm{n}_3\omega^2\tag{16}$$

$$\kappa_2=3\mathrm{n}_4+2\mathrm{n}_5,\kappa_3=\mathrm{n}_3\tag{17}$$

$$\kappa_4=-k\mathrm{n}_0-\mathrm{n}_1\omega^2-\mathrm{n}_3\omega^3\tag{18}$$

and

$$\kappa_5=\mathrm{n}_2+\mathrm{n}_4\omega,\kappa_6=\mathrm{n}_1+3\mathrm{n}_3\omega\tag{19}$$

When α is a pure complex number i.e. α=iμ, Eq. (15) becomes

$$i[\mu\kappa_1\overline{E}_\xi+\mu\kappa_2\overline{E}^2\overline{E}_\xi+(-\mu^3)\kappa_3\overline{E}_{\xi\xi\xi}+\kappa_4\overline{E}+\kappa_5\overline{E}^3+(-\mu^2)\kappa_6\overline{E}_{\xi\xi}]=0\tag{20}$$

$\overline{E}$ been a real number, it is easy to see that a solution of Eq. (17) is also a solution of Eq. (15) when the following transformations are considered:

$$\alpha\leftarrow\mu,\alpha^2\leftarrow-\mu^2,\alpha^3\leftarrow-\mu^3\tag{21}$$

Thus, from Eq. (15), leading to the hyperbolic function solutions of the GHNLS, we can deduce their corresponding PTWS without solve it again.

Inserting Eq. (13) into Eq.(15), and separating in real part and imaginary part, we obtain the main equations which will be in the center of all analysis,

$$\begin{aligned}
&\alpha\kappa_2 a^3\left(m\sinh^{3m-1}\xi\sec h^{3n-1}\xi-n\sinh^{3m+1}\xi\sec h^{3n+1}\xi\right)+\alpha\kappa_2 qb^3\left(q\sinh^{3q-1}\xi\sec h^{3r-1}\xi-r\sinh^{3q+1}\xi\sec h^{3r+1}\xi\right)\\
&+ab^2\alpha\kappa_2[(m+2q)\sinh^{m+2q-1}\xi\sec h^{n+2r-1}\xi-(n+2r)\sinh^{m+2q+1}\xi\sec h^{n+2r+1}\xi]+a^2b\alpha\kappa_2[(q+2m)\sinh^{2m+q-1}\xi\sec h^{2n+r-1}\xi...\\
&-(r+2n)\sinh^{2m+q+1}\xi\sec h^{2n+r+1}\xi]-a\alpha m\begin{pmatrix}2\kappa_3\alpha^2+3\alpha^2mn\kappa_3\\-3\alpha^2\kappa_3 m-\kappa_1\end{pmatrix}\sinh\xi^{m-1}\sec h\xi^{n-1}+a\alpha n\begin{pmatrix}3\alpha^2\kappa_3 n+3\alpha^2mn\kappa_3\\+2\kappa_3\alpha^2-\kappa_1\end{pmatrix}\sinh\xi^{m+1}\sec h\xi^{n+1}\\
&-\alpha bq\begin{pmatrix}2\kappa_3\alpha^2+3\alpha^2qr\kappa_3\\-3\alpha^2\kappa_3 q-\kappa_1\end{pmatrix}\sinh\xi^{q-1}\sec h\xi^{r-1}+\alpha r(2\kappa_3\alpha^2+3\alpha^2qr\kappa_3+3\alpha^2\kappa_3 r-\kappa_1)\sinh\xi^{q+1}\sec h\xi^{r+1}\\
&+\alpha^3\kappa_3 a[m(m-1)(m-2)\sinh^{m-3}\xi\sec h^{n-3}\xi-n(n-1)(n-2)\sinh^{m+3}\xi\sec h^{n+3}\xi]\\
&+\alpha^3\kappa_3 b[q(q-1)(q-2)\sinh^{q-3}\xi\sec h^{r-3}\xi-r(r-1)(r-2)\sinh^{q+3}\xi\sec h^{r+3}\xi]=0
\end{aligned}\tag{22}$$

and

$$\begin{aligned}
&3\kappa_5 a^2b\sinh^{2m+q}\xi\sec h^{2n+r}\xi+3\kappa_5 ab^2\sinh^{2q+m}\xi\sec h^{2r+m}\xi+\kappa_5 a^3\sinh^{3m}\xi\sec h^{3n}\xi+\kappa_5 b^3\sinh^{3q}\xi\sec h^{3r}\xi\\
&-a\begin{pmatrix}-\kappa_4+\alpha^2\kappa_6 n\\+2\alpha^2\kappa_6 mn-\alpha^2\kappa_6 m\end{pmatrix}\sinh^m\xi\sec h^n\xi-b\begin{pmatrix}-\kappa_4+\alpha^2\kappa_6 r\\+2\alpha^2\kappa_6 qr-\alpha^2\kappa_6 q\end{pmatrix}\sinh^q\xi\sec h^r\xi\\
&+\alpha^2\kappa_6 a\begin{bmatrix}m(m\text{-}1)\sinh^{m-2}\xi\sec h^{n-2}\xi\\+n(n+1)\sinh^{m+2}\xi\sec h^{n+2}\xi\end{bmatrix}+\alpha^2\kappa_6 b\begin{bmatrix}q(q\text{-}1)\sinh^{q-2}\xi\sec h^{r-2}\xi\\+r(r+1)\sinh^{q+2}\xi\sec h^{r+2}\xi\end{bmatrix}=0
\end{aligned}\tag{23}$$

Equations (19) are the range equations of the coefficients from which we study all the possibilities of obtaining solutions.

4. Analysis of the Range Equations (22) and (23) and Solutions

The search for the terms of the equations (13) and (14) that are grouped allows to obtain the values of m, n, q and r for which it is possible to obtain solutions. So, fo $m=n=0$ and $q=r$, some terms of Eqs. (22) and (23) merge for $q=r=\{0,\pm1,\pm2,\pm3,\pm4\}$.

a) Form $n=q=r=0$, Eqs. (22) and (23) reduce to

$$(a+b)\left[\kappa_5(a+b)^2+\kappa_4\right]=0\tag{24}$$

and the constant solution of Eq. (2) is

$$E(x,t)=\pm\sqrt{\frac{-\kappa_4}{\kappa_5}}=\pm\sqrt{\frac{\mathrm{kn}_0+\mathrm{n}_1\omega^2+\mathrm{n}_3\omega^3}{\mathrm{n}_2+\mathrm{n}_4\omega}}=cst\tag{25}$$

b) For m=n=0 and q=r= ±1 Eqs. (19) become

$$\alpha b(4\alpha^2\kappa_3+\kappa_2b^2+\kappa_2a^2+\kappa_1)\sec h^2\xi-\alpha b(\kappa_2b^2+6\alpha^2\kappa_3)\sec h^4\xi+2\alpha\kappa_2ab^2\sinh\xi\sec h^3\xi=0\tag{26}$$

and

$$a(\kappa_5a^2+3\kappa_5b^2+\kappa_4)-3\kappa_5ab^2\sec h^2\xi+b(\kappa_4+3\kappa_5a^2+\kappa_5b^2)\sinh\xi\sec h\xi-b(\kappa_5b^2+2\alpha^2\kappa_6)\sinh\xi\sec h^3\xi=0\tag{27}$$

The following coefficient equations are obtained from Eqs. (26) and (27):

$$4\alpha^2\kappa_3+\kappa_2b^2+\kappa_2a^2+\kappa_1=0\tag{28}$$

$$\kappa_2b^2+6\alpha^2\kappa_3=0\tag{29}$$

$$a(\kappa_5a^2+3\kappa_5b^2+\kappa_4)=0\tag{30}$$

$$2\alpha\kappa_2ab^2=0\tag{31}$$

$$\kappa_5ab^2=0\tag{32}$$

$$b(\kappa_4+3\kappa_5a^2+\kappa_5b^2)=0\tag{33}$$

and

$$b(\kappa_5b^2+2\alpha^2\kappa_6)=0\tag{34}$$

The resolution of this algebraic system, taking into account Eqs. (16),..,(19) permits to obtain the parameters

$$a=0,\ b=\pm\sqrt{\frac{-6\mathrm{n}_3}{3\mathrm{n}_4+2\mathrm{n}_5}}\alpha\tag{35}$$

with $-\mathrm{n}_3(3\mathrm{n}_4+2\mathrm{n}_5)>0$,

$$\omega=\frac{3\mathrm{n}_2\mathrm{n}_3-\mathrm{n}_1(3\mathrm{n}_4+2\mathrm{n}_5)}{6\mathrm{n}_3(\mathrm{n}_4+\mathrm{n}_5)}\tag{36}$$

with $n_3(n_4+n_5)\neq 0$,

$$k=\frac{\begin{pmatrix}-2\alpha^2 n_1-6\alpha^2 n_3\omega\\ -\omega^2 n_1-n_3\omega^3\end{pmatrix}}{n_0},\ n_0\neq 0 \quad (37)$$

$$\beta=\frac{\begin{pmatrix}-2\alpha^2 n_3-2n_1\omega\\ -3n_3\omega^2\end{pmatrix}}{n_0},\ n_0\neq 0 \quad (38)$$

Inserting the values of ω in Eqs. (37) and (38), k and β can be rewritten as

$$k=\frac{\begin{bmatrix}216\alpha^2 n_1 n_3^2 n_4(n_4-n_5)^2\\ -648\alpha^2 n_2 n_3^2(n_4+n_5)^2-27(n_2n_3+n_1n_4)\times\\ (n_1n_4-n_2n_3)^2-72n_1^2 n_4 n_5(n_1n_4-n_2n_2)\\ -4n_1^2n_5^2(4n_1n_5-9n_2n_3+15n_1n_4)\end{bmatrix}}{216n_3^2(n_4+n_5)^3} \quad (39)$$

$$\beta=\frac{\begin{bmatrix}-24\alpha^2 n_3^2(n_4+n_5)^2+4n_1^2n_5(n_5+2n_4)\\ +3(n_1n_4+3n_2n_3)(n_1n_4-n_2n_3)\end{bmatrix}}{12n_0 n_3(n_4+n_5)^2} \quad (40)$$

The optical dark soliton solution and the travelling wave solution of the GHNLS equation in this condition read

$$E(z,t)=\pm\sqrt{\frac{-6n_3}{3n_4+2n_5}}\alpha\tanh\alpha\left(t-\frac{\begin{pmatrix}-2\alpha^2n_3-2n_1\omega\\ -3n_3\omega^2\end{pmatrix}}{n_0}z\right)\times\exp-i\left(\begin{matrix}\frac{-2\alpha^2n_1-6\alpha^2n_3\omega-\omega^2n_1-n_3\omega^3}{n_0}z\\ -\frac{3n_2n_3-n_1(3n_4+2n_5)}{6n_3(n_4+n_5)}t\end{matrix}\right) \quad (41)$$

and

$$E(z,t)=\pm\sqrt{\frac{-6n_3}{3n_4+2n_5}}\alpha\cot an\mathrm{h}\,\alpha\left(t-\frac{\begin{matrix}-2\alpha^2n_3-2n_1\omega\\ -3n_3\omega^2\end{matrix}}{n_0}z\right)\times\exp-i\left(\begin{matrix}\frac{-2\alpha^2n_1-6\alpha^2n_3\omega-\omega^2n_1-n_3\omega^3}{n_0}z\\ -\frac{3n_2n_3-n_1(3n_4+2n_5)}{6n_3(n_4+n_5)}t\end{matrix}\right) \quad (42)$$

The corresponding TPWS are

$$E(z,t)=\pm i\sqrt{\frac{-6n_3}{3n_4+2n_5}}\mu\tan\mu\left(t-\frac{\begin{matrix}2\mu^2n_3-\\ 2n_1\omega-3n_3\omega^2\end{matrix}}{n_0}z\right)\times\exp-i\left(\begin{matrix}\frac{2\mu^2n_1+6\mu^2n_3\omega-\omega^2n_1-n_3\omega^3}{n_0}z\\ -\frac{3n_2n_3-n_1(3n_4+2n_5)}{6n_3(n_4+n_5)}t\end{matrix}\right) \quad (43)$$

and

$$E(z,t)=\pm i\sqrt{\frac{-6n_3}{3n_4+2n_5}}\mu\cot an\mu\left(t-\frac{\begin{matrix}2\mu^2n_3+\\ 2n_1\omega-3n_3\omega^2\end{matrix}}{n_0}z\right)\times\exp-i\left(\begin{matrix}\frac{2\mu^2n_1+6\mu^2n_3\omega-\omega^2n_1-n_3\omega^3}{n_0}z\\ -\frac{3n_2n_3-n_1(3n_4+2n_5)}{6n_3(n_4+n_5)}t\end{matrix}\right) \quad (44)$$

No solution is found for $m=n=q=r$ and $|r|\geq 2$.

c) For $m=0,\ n=0,\ q=0$ and $r=1$, Eqs. (19), become

$$-\alpha b(\kappa_1+\kappa_2a^2+\alpha^2\kappa_3)\sinh\xi\,\mathrm{sech}^2\xi-2\alpha\kappa_2ab^2\sinh\xi\,\mathrm{sech}^3\xi+\alpha b(-\kappa_2b^2+6\alpha^2\kappa_3)\sinh\xi\,\mathrm{sech}^4\xi=0 \quad (45)$$

and

$$i[a(\kappa_4+\kappa_5a^2)+(\kappa_4b+3\kappa_5a^2b+\alpha^2\kappa_6b)\mathrm{sech}\xi+3\kappa_5ab^2\mathrm{sech}^2\xi+(\kappa_5b^3-2\alpha^2\kappa_6b)\mathrm{sech}^3\xi]=0 \quad (46)$$

The resolution of the algebraic system resulting from this equation gives

$$a = 0, b = \pm\sqrt{\frac{6n_3}{3n_4 + 2n_5}}\alpha \tag{47}$$

$$\omega = \frac{3n_2n_3 - 3n_1n_4 - 2n_1n_5}{6n_3(n_4 + n_5)} \tag{48}$$

$$k = (\alpha^2 n_1 + 3\alpha^2 n_3\omega - n_1\omega^2 - n_3\omega^3) / n_0 \tag{49}$$

and

$$\beta = (\alpha^2 n_3 - 2n_1\omega - 3n_3\omega^2) / n_0 \tag{50}$$

Inserting the value of ω in Eq. (35), k and β are express as follow

$$k = -\frac{\begin{bmatrix} 108\alpha^2 n_1 n_3^2 n_4 (n_4 - n_5)^2 \\ -324\alpha^2 n_2 n_3^2 (n_4 + n_5)^2 + 27(n_2n_3 + n_1n_4)\times \\ (n_1n_4 - n_2n_3)^2 + 72 n_1^2 n_4 n_5 (n_1n_4 - n_2n_2) \\ +4 n_1^2 n_5^2 (4n_1n_5 - 9n_2n_3 + 15n_1n_4) \end{bmatrix}}{216 n_3^2 (n_4 + n_5)^3} \tag{51}$$

and

$$\beta = \frac{\begin{bmatrix} 12\alpha^2 n_3^2 (n_4 + n_5)^2 + 4n_1^2 n_5 (n_5 + 2n_4) \\ +3(n_1n_4 + 3n_2n_3)(n_1n_4 - n_2n_3) \end{bmatrix}}{12 n_0 n_3 (n_4 + n_5)^2} \tag{52}$$

Thus, the optical bright soliton obtained here which its equivalent PTWS read

$$E(z,t) = \pm\sqrt{\frac{6n_3}{3n_4 + 2n_5}}\alpha \operatorname{sech}\alpha\left(t - \frac{\alpha^2 n_3 - 2n_1\omega - 3n_3\omega^2}{n_0}z\right)\times\exp - i\left(\frac{\alpha^2 n_1 + 3\alpha^2 n_3\omega - n_1\omega^2 - n_3\omega^3}{n_0}z - \frac{3n_2n_3 - 3n_1n_4 - 2n_1n_5}{6n_3(n_4 + n_5)}t\right) \tag{53}$$

and

$$E(z,t) = \pm\sqrt{\frac{-6n_3}{3n_4 + 2n_5}}\mu \operatorname{cosec}\mu\left(t - \frac{-\mu^2 n_3 - 2n_1\omega - 3n_3\omega^2}{n_0}z\right)\times\exp - i\left(\frac{-\mu^2 n_1 - 3\mu^2 n_3\omega - n_1\omega^2 - n_3\omega^3}{n_0}z - \frac{3n_2n_3 - 3n_1n_4 - 2n_1n_5}{6n_3(n_4 + n_5)}t\right) \tag{54}$$

The new constraint relations susceptible to make propagate dark and bright optical solitons in the nonlinear fiber are $n_3(3n_4 + 2n_5) < 0$ and $n_3(3n_4 + 2n_5) > 0$ respectively. From these conditions, we can see that optical dark and bright solitons can propagate both in anomalous dispersion regime and normal dispersion regime. The changing of these results is due to the presence of higher-nonlinear effects and the TOD.

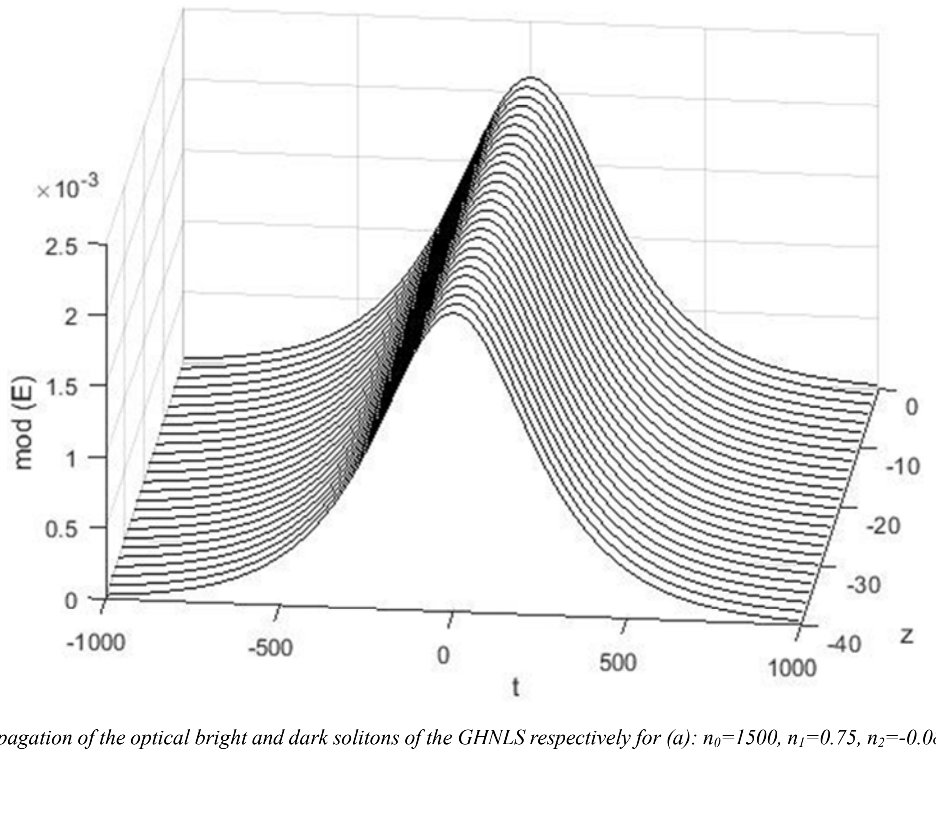

Figure 1. *Numerical propagation of the optical bright and dark solitons of the GHNLS respectively for (a): n_0=1500, n_1=0.75, n_2=-0.08556, n_3=-0.059, n_4=0.5, n_5=0.25, α=0.06.*

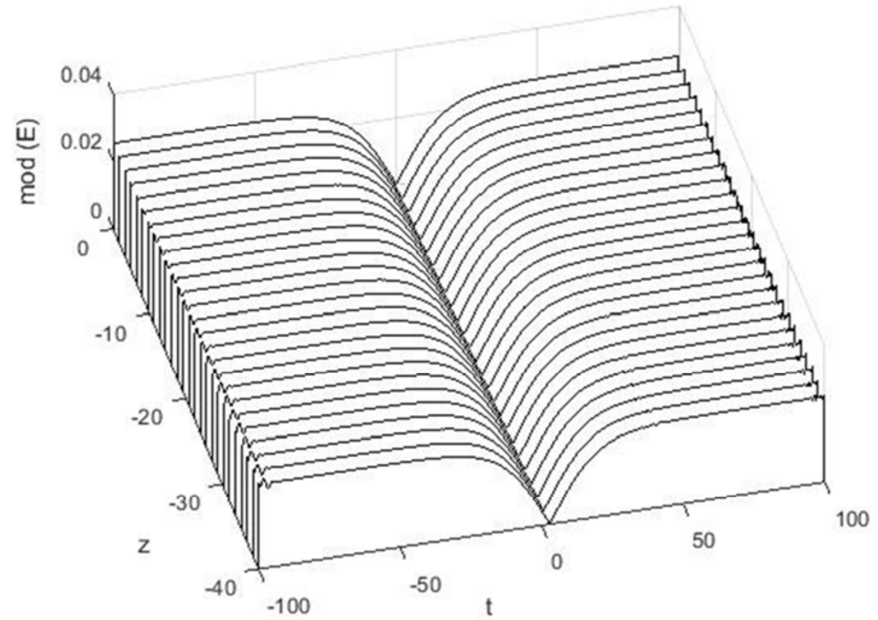

Figure 2. *Numerical propagation of the optical bright and dark solitons of the GHNLS respectively for n_0=3000, n_1=0.75, n_2=-0.08556, n_3=-0.059, n_4=0.5, n_5=0.25, α=0.05.*

Using the one-soliton ansatz, H. Kumar and F. Chand derive optical dark and bright soliton solutions, corresponding to previous solutions for n_0=1. They also derive periodic solutions like dn, ns, ds, and cs functions and show that these Jacobi elliptic functions can degenerate into trigonometric functions, i.e., sn(ξ)→sin(ξ), cn(ξ)→cos(ξ), dn(ξ)→1, and the periodic traveling wave solutions could become the periodic trigonometric solutions which correspond to weak localization [33]. Figures1 and 2 show the numerical propagation of these different optical solitons.

The numerical simulation is obtained using the fourth order Runge-Kutta method and integrating factor.

d) For $m=n=q=0, r=-1$, the computation of the resulting algebraic system gives

$$\omega=-\frac{n_2}{n_4} \tag{55}$$

$$k=\frac{\alpha^2 n_4^{\,2}(n_1n_4-3n_2n_3)-n_2^{\,2}(n_1n_4-n_2n_3)}{n_0n_4^{\,3}} \tag{56}$$

$$\beta=\frac{\alpha^2 n_3n_4^{\,2}+2n_1n_2n_4-3n_2^{\,2}n_3}{n_0n_4^{\,2}} \tag{57}$$

when $a=0$, with the constraint equation $2n_5+3n_4=0$.

When $a\neq0$, the constraint equations are $n_1n_4-3n_2n_3=0$, $2n_5+3n_4=0$ and the constants

$$\omega=-\frac{n_2}{n_4}, k=-\frac{2n_1^{\,3}}{27n_0n_3^{\,2}}, \beta=\frac{3\alpha^2n_3^{\,2}+n_1^{\,2}}{3n_0n_3} \tag{58}$$

Using the transformations Eq. (21), the obtained PTWS here read

$$E(z,t)=b\cos\mu\left(t-\frac{-\mu^2n_3n_4^{\,2}+2n_1n_2n_4-3n_2^{\,2}n_3}{n_0n_4^{\,2}}z\right)\times\exp-i\left(\frac{-\mu^2n_4^{\,2}(n_1n_4-3n_2n_3)-n_2^{\,2}(n_1n_4-n_2n_3)}{n_0n_4^{\,3}}z+\frac{n_2}{n_4}t\right) \tag{59}$$

and

$$E(z,t)=\left[a+b\cos\mu\left(t-\frac{-3\mu^2n_3^{\,2}+n_1^{\,2}}{3n_0n_3}z\right)\right]\times\exp-i\left(-\frac{2n_1^{\,3}}{27n_0n_3^{\,2}}z+\frac{n_2}{n_4}t\right) \tag{60}$$

e) For $m=n=q=0, r=-2$, we found

$$\omega=-\frac{n_2}{n_4}, k=-\frac{2n_1^{\,3}}{27n_0n_3^{\,2}}, \beta=\frac{4\alpha^2n_3^{\,2}+n_1^{\,2}}{3n_0n_3} \tag{61}$$

with the parametric equations $2n_5+3n_4=0$, and $n_1n_4-3n_2n_3=0$.

Under these parametric constrains, solution is written

$$E(z,t)=\left[a+b\cos^2\mu(t-\frac{-4\mu^2 n_3{}^2+n_1{}^2}{3n_0n_3}z)\right]\times\exp-i\left(-\frac{2n_1{}^3}{27n_0n_3{}^2}z+\frac{n_2}{n_4}t\right) \tag{62}$$

f) For $m=n=0, q=1, r=0$. The constants ω, k, and β are given by

$$\omega=-\frac{n_2}{n_4} \tag{63}$$

$$k=\frac{\begin{bmatrix}\alpha^2 n_4{}^2(n_1n_4-3n_2n_3)\\ -n_2{}^2(n_1n_4-n_2n_3)\end{bmatrix}}{n_0n_4{}^3} \tag{64}$$

$$\beta=\frac{\alpha^2 n_3n_4{}^2+2n_1n_2n_4-3n_2{}^2n_3}{n_0n_4{}^2} \tag{65}$$

with parametric constraint $2n_5+3n_4=0$, when a=0.
For a≠0, constantsarewritten

$$\omega=-\frac{n_2}{n_4}, k=-\frac{2n_1{}^3}{27n_0n_3{}^2}, \beta=\frac{3\alpha^2 n_3{}^2+n_1{}^2}{3n_0n_3} \tag{66}$$

with the constraint equations $2n_5+3n_4=0$,
and $n_1n_4-3n_2n_3=0$.
The different solutions of the GHNLS reported here are

$$E(z,t)=ib\sin\mu\left(t-\frac{\begin{matrix}-\mu^2 n_3n_4{}^2+2n_1n_2n_4\\ -3n_2{}^2n_3\end{matrix}}{n_0n_4{}^2}z\right)\times\exp-i\left[\frac{\begin{matrix}-\mu^2 n_4{}^2(n_1n_4-3n_2n_3)\\ -n_2{}^2(n_1n_4-n_2n_3)\end{matrix}}{n_0n_4{}^3}z+\frac{n_2}{n_4}t\right] \tag{67}$$

and

$$E(z,t)=\left[a+ib\sin\mu(t-\frac{-3\mu^2 n_3{}^2+n_1{}^2}{3n_0n_3}z)\right]\times\exp-i\left(-\frac{2n_1{}^3}{27n_0n_3{}^2}z+\frac{n_2}{n_4}t\right) \tag{68}$$

g) $m=n=0, q=1, r=-1$ Following the same procedure, we find the constants

$$\omega=-\frac{n_2}{n_4} \tag{69}$$

$$k=\frac{\begin{pmatrix}\alpha^2 n_4{}^2(n_1n_4-3n_2n_3)\\ -n_2{}^2(n_1n_4-n_2n_3)\end{pmatrix}}{n_0n_4{}^3} \tag{70}$$

$$\beta=\frac{\alpha^2 n_3n_4{}^2+2n_1n_2n_4-3n_2{}^2n_3}{n_0n_4{}^2} \tag{71}$$

and constraint equation $2n_5+3n_4=0$, when a=0,
and

$$\omega=-\frac{n_2}{n_4}, k=-\frac{2n_1^3}{27n_0n_3^2}, \beta=\frac{4\alpha^2n_3^2+n_1^2}{3n_0n_3} \tag{72}$$

when the parameters n_i also satisfy the constraints $2n_5+3n_4=0$ and $n_1n_4-3n_2n_3=0$ when $a\neq 0$.

The PTWS of the GHNLS read

$$E(z,t)=i\frac{b}{2}\sin 2\mu\left(t-\frac{\begin{array}{c}-\mu^2n_3n_4^2\\ +2n_1n_2n_4-3n_2^2n_3\end{array}}{n_0n_4^2}z\right)\times\exp-i\left[\frac{\begin{array}{c}-\mu^2n_4^2(n_1n_4-3n_2n_3)\\ -n_2^2(n_1n_4-n_2n_3)\end{array}}{n_0n_4^3}z+\frac{n_2}{n_4}t\right] \tag{73}$$

and

$$E(z,t)=\left[a+i\frac{b}{2}\sin 2\mu(t-\frac{-4\mu^2n_3^2+n_1^2}{3n_0n_3}z)\right]\times\exp-i\left(-\frac{2n_1^3}{27n_0n_3^2}z+\frac{n_2}{n_4}t\right) \tag{74}$$

For $m=n=0$, $q=2$, $r=0$, the following TPWSs are obtained

$$E(z,t)=-b\sin^2\mu(t-\frac{-4\mu^2n_3^2+n_1^2}{3n_0n_3}z)\times\exp-i\left(-\frac{2n_1^3}{27n_0n_3^2}z+\frac{n_2}{n_4}t\right) \tag{75}$$

and

$$E(z,t)=\left\{a-2a\sin^2\mu\left(t-\frac{\begin{array}{c}-4\mu^2n_3n_4^2+\\ 2n_1n_2n_4-3n_2^2n_3\end{array}}{n_0n_4^2}z\right)\right\}\times\exp-i\left[\frac{\begin{array}{c}-4\mu^2n_4^2(n_1n_4-3n_2n_3)\\ -n_2^2(n_1n_4-n_2n_3)\end{array}}{n_0n_4^3}z+\frac{n_2}{n_4}t\right] \tag{76}$$

For $m=1, n=0, q=0, r=-1$, setting the coefficients of each hyperbolic function of the main equations Eqs. (22) and (23) to zero, and Computing the resulting algebraic system, coefficients allowing to write solution is obtained such as

$$E(z,t)=\left[\begin{array}{c}ia\sin\mu(t-\frac{-3\mu^2n_3^2+n_1^2}{3n_0n_3}z)\\ +b\cos\mu(t-\frac{-3\mu^2n_3^2+n_1^2}{3n_0n_3}z)\end{array}\right]\times\exp-i\left(-\frac{2n_1^3}{27n_0n_3^2}z+\frac{n_2}{n_4}t\right) \tag{77}$$

No solution is found for $m=0, n=1, q=1$ and $r=1$.

The different constraints on the parameters of the equation also allow to define Schrodinger equations family with strong nonlinearity, and their exact solutions immediately.

When $n_1n_4-3n_2n_3\neq 0$ and $2n_5+3n_4=0$ for example, the GHNLS equation of eq.(2) lead to the following HNLS equation family

$$n_0E_z+in_1E_{tt}+in_2|E|^2E+\frac{n_1n_4}{3n_2}E_{ttt}+n_4\left(|E|^2E\right)_t-\frac{3n_4}{2}\left(|E|^2\right)_tE=0 \tag{78}$$

From a purely mathematical point of view, it is now possible to propose an infinity of strongly non-linear differential equations of the Schrodinger type, describing the propagation dynamics of solitary waves in optical fibers admitting exact solutions.

The procedure for obtaining these equations consists in assigning the values to the coefficients n_i($i=0,1,2,3,4,5$) through the different constraint equations that bind them, and thus deduce from Eq. (78) the corresponding NPDEs as well as their exact solutions given by Eqs. (59, 60, …, 77).

Considering for example, the values $n_0=1, n_1=3, n_2=2, n_4=2$,we obtain from the constraint equation coefficients $(n_1n_4-3n_2n_3=0, 2n_5+3n_4=0)$ and $n_3=1$ and $n_5=-3$. One of the solutions of Eq.(78) given by Eq.(60) reads

$$E(z,t)=\left\{a+b\cos\mu[t-(-\mu^2+3)z]\right\}\times\exp-i\left(-2z+t\right) \tag{79}$$

Other wise, when the coefficients $(n_0=1, n_1=3, n_2=1, \quad n_3=2, n_4=2, n_5=1)$ with $\alpha=1$ are chosen respecting the constraints $n_3\left(2n_5+3n_4\right)\succ 0,\ \ n_0\neq 0,\ \ n_3\left(n_5+n_4\right)\neq 0,$ Eq.(2) becomes

$$E_z+3iE_{tt}+i|E|^2E+2E_{ttt}+2\left(|E|^2E\right)_t+\left(|E|^2\right)_tE=0 \tag{80}$$

Based on these values, the pulsed solitary wave solution of Eq. (80) given by Eq. (53), is written

$$E(z,t)=\pm\sqrt{\frac{3}{2}}\,\text{sech}\left(t-\frac{7}{2}z\right)\times\exp-i\left(-\frac{1}{2}z+\frac{1}{2}t\right) \tag{81}$$

5. Conclusion

We have established in this work the conditions to obtain all partial differential equations of the same family as the Schrodinger nonlinear equation describing the dynamic of the propagation of a soliton wave in a strongly nonlinear optical fiber. In practical terms, the results obtained can be part of a framework in which the properties or components of the optical fiber are modified so as to choose a solitary wave or any type of signal that can propagate in the transmission medium thus constituted, with months of dispersion and dissipation, simply because the coefficients related to these terms have been judiciously selected.

In a purely mathematical sense, this way of investigating strongly nonlinear Schrödinger equations allows not only to determine the exact solutions, but also new partial differential equations of the same family at the same time with their exact solutions.

To bring closer to physical reality in a practical sense, we have used numerical simulations to verify the propagation of some solutions obtained. The propagation of some solutions obtained reassures when to the practical and experimental application of our study and our obtained results.

The depth of this work lies in the possibility for engineers specialized in the construction of transmission media and more specifically optical fibers, to set up new optical fibers whose properties are adapted to the types of signals that we want to propagate, among which solitary waves.

References

[1] D. Anderson, (1983). Variational approach to nonlinear pulse propagation in optical fibers. Physical review A, 27, 3135.

[2] P. Emplit, J. P. Hamaide, F. Reynaud, C. Froehly, A. Barthelemy, (1987). Picosecond steps and dark pulses through nonlinear single mode fibers. Optics Communications, 62, 374-379.

[3] B. Tian, W. R. Shan, C. Y. Zhang, G. M. Wei, Y. T. Gao, (2005). Transformations for a generalized variable-coefficient nonlinear Schrödinger model from plasma physics, arterial mechanics and optical fibers with symbolic computation. The European Physical Journal B-Condensed Matter and Complex Systems, 47, 329-332.

[4] C. A. Jones, P. H. Roberts, (1982). Motions in a Bose condensate. IV. Axisymmetric solitary waves. Journal of Physics A: Mathematical and General, 15, 2599.

[5] P. Ruprecht, M. J. Holland, K. Burnett, M. Edwards, (1995). Time-dependent solution of the nonlinear Schrödinger equation for Bose-condensed trapped neutral atoms. Physical Review A, 51, 4704.

[6] H. B. Thacker, (1981). Exact integrability in quantum field theory and statistical systems, Reviews of Modern Physics, 53,253.

[7] M. Peyrard, (2004).Nonlinear dynamics and statistical physics of DNA. Nonlinearity, 17, 2 R1.

[8] C. Nore, M. E. Brachet, S. Fauve, (1993). Numerical study of hydrodynamics using the nonlinear Schrödinger equation. Physica D: Nonlinear Phenomena, 65,154-162.

[9] A. Hasegawa, F. Tappert, (1973). Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion. Applied Physics Letters, 23,142-144.

[10] A. Hasegawa, F. Tappert, (1973).Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. II. Normal dispersion. Applied Physics Letters, 23, 171-172.

[11] Z. Xu, L. Li, Z. Li, Z, G. Zhou, (2002). Soliton interaction under the influence of higher-order effects. Optics Communications, 210, 375-384.

[12] Z. Li, L. Li, H. Tian, G. Zhou, (2000). New types of solitary wave solutions for the higher order nonlinear Schrödinger equation. Physical review letters, 84, 4096.

[13] K. Porsezian, K. Nakkeeran, (1996). Optical solitons in presence of Kerr dispersion and self-frequency shift. Physical review letters, 76, 3955.

[14] C. Sien, C. W. Chang, S. Wen, (1994). Femtosecond soliton propagation in erbium-doped fiber amplifiers: the equivalence of two different models. Optics communications, 106,193-196.

[15] S. L. Palacios, A. Guinea, J. M. Fernandez-Diaz, R. D. Crespo, (1999). Dark solitary waves in the nonlinear Schrödinger equation with third order dispersion, self-steepening, and self-frequency shift. Physical Review E, 60, R45

[16] D. Mihalache, N. Truta, L-C. Crasovan, (1997). Painlevé analysis and bright solitary waves of the higher-order nonlinear Schrödinger equation containing third-order dispersion and self-steepening term. Physical Review E, 56, 1064.

[17] J. R. Bogning, (2014). Solitary Wave Solutions of the High-order Nonlinear Schrödinger Equation in Dispersive Single Mode Optical Fibers. American Journal of Computational and Applied Mathematics, 4, 45-50.

[18] N. Sasa, J. Satsuma, (1991). New-type of soliton solutions for a higher-order nonlinear Schrödinger equation. Journal of the Physical Society of Japan, 60, 409-417.

[19] V. I. Karpman, (2004). The extended third-order nonlinear Schrödinger equation and Galilean transformation. The European Physical Journal B-Condensed Matter and Complex Systems, 39,341-350.

[20] S. Ghosh, A. Kundu, S. Nandy, (1999). Soliton solutions, Liouville integrability and gauge equivalence of Sasa Satsuma equation. Journal of Mathematical Physics, 40, 1993-2000.

[21] J. Xu, F. Engui, (2013). The unified transform method for the Sasa–Satsuma equation on the half-line, Proc. R. Soc. A, 469,20130068.

[22] Wang, Mingliang, L. Xiangzheng, Jinliang Zhang, (2007).Sub-ODE method and solitary wave solutions for higher order nonlinear Schrödinger equation. Physics Letters A, 363, 96-101.

[23] Yan, Zhenya, (2003). Generalized method and its application in the higher-order nonlinear Schrodinger equation in nonlinear optical fibres. Chaos, Solitons & Fractals, 16,759-766.

[24] Hong, Baojian, L. Dianchen, (2009). New Jacobi elliptic functions solutions for the higher-order nonlinear Schrodinger equation, International journal of nonlinear science, 7, 360-367.

[25] T. C. T. Djeumen, J. R. Bogning, T. C. Kofane, (2011).Multi-Soliton solutions of the modified Kuramoto-Sivashinsky's equation by the BDK method. Far East Journal of dynamical systems, 15, 83-98.

[26] T. C. T. Djeumen, J. R. Bogning, T. C. Kofané, (2012).Modulated Soliton Solution of the Modified Kuramoto-Sivashinsky's Equation. American Journal of Computational and Applied Mathematics, 2, 218-224.

[27] J. R. Bogning, T. C. T. Djeumen, T. C. Kofané, (2012). Construction of the soliton solutions of the Ginzburg–Landau equations by the new Bogning–Djeumen Tchaho–Kofané method. Physica Scripta, 85, 025013.

[28] J. R. Bogning, (2012). Generalization of the Bogning-Djeumen Tchaho-Kofané method for the construction of the solitary waves and the survey of the instabilities. Far East J. Dyn. Sys, 20,101-119.

[29] J. R. Bogning, T. C. T. Djeumen, T. C. Kofané, (2013).Solitary wave solutions of the modified Sasa-Satsuma nonlinear partial differential equation. American Journal of Computational and Applied Mathematics, 3, 97-107.

[30] J. R. Bogning, (2013). Pulse soliton solutions of the modified KdV and Born-Infeld equations. International Journal of Modern Nonlinear Theory and Application, 2, 135.

[31] J. R. Bogning, (2015). N th Order Pulse Solitary Wave Solution and Modulational Instability in the Boussinesq Equation. American Journal of Computational and Applied Mathematics, 5,182-188.

[32] J. R. Bogning, C. T. Djeumen, H. M. Omanda, (2016).Combined Solitary Wave Solutions in Higher-order Effects Optical Fibers. British Journal of Mathematics & Computer Science, 13, 1-12.

[33] H. Kumar, C. Fakir, (2013).Dark and bright solitary wave solutions of the higher order nonlinear Schrödinger equation with self-steepening and self-frequency shift effects. Journal of Nonlinear Optical Physics & Materials, 22, 1350001.

[34] R. Njikue, J. R. Bogning, T. C. Kofane, (2018).Exact bright and dark solitary wave solutions of the generalized higher-order nonlinear Schrödinger equation describing the propagation of ultra-short pulse in optical fiber. Journal of Physics Communications, 2, 025030.

[35] A. R. Seadawy, L. Dianchen, (2017).Bright and dark solitary wave soliton solutions for the generalized higher order nonlinear Schrödinger equation and its stability. Results in physics, 7, 43-48.

[36] K. Porsezian, K. Senthilnathan, S. Devipriya, (2005). Modulational instability in fiber Bragg grating with non-Kerr nonlinearity. IEEE journal of quantum electroics, 41,789-796.